Stochastic approximations with operator means

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Introduction

In this talk, E will denote a Hilbert space

- $\mathbb{S}(E)$ denote the real Banach space of bounded, linear, self-adjoint operators over E
- $\mathbb{P}(E) \subseteq \mathbb{S}(E)$ denotes the cone of invertible positive definite and $\hat{\mathbb{P}}(E)$ the cone of positive (semi-definite) operators

$\mathbb{S}$ and $\mathbb{P}$ are partially ordered cones with the positive definite, also called Loewner order:

$$A \leq B \text{ iff } B - A \text{ is positive semidefinite, i.e.}$$

$$v^*(B - A)v \geq 0 \text{ for all } v \in E$$

The multivariable Karcher and Wasserstein means

$\omega = (w_1, \ldots, w_n) \in \Delta_n$ probability vector, $w_i > 0$, $\sum_{i=1}^n w_i = 1$,
$\mathbb{A} = (A_1, \ldots, A_n)$, $A_i \in \mathbb{P}$

$$\Lambda(\omega; \mathbb{A}) := \operatorname*{arg\,min}_{X \in \mathbb{P}} \sum_{i=1}^{n} w_i d^2(X, A_i)$$

where

- $d^2(X, A) = \operatorname{tr}\{\log^2(X^{-1/2} A X^{-1/2})\}$ for the Karcher mean, $X \mapsto d^2(X, A)$ is a strictly geodesically convex function by the NPC property of the trace metric;
- $d^2(A, B) = \operatorname{tr}(A + B) - 2 \operatorname{tr}(A^{1/2} B A^{1/2})^{1/2}$ for the Bures-Wasserstein mean, $X \mapsto d^2(X, A)$ is a strictly convex function with respect to the Euclidean structure (Hilbert-Schmidt metric)

The infinite dimensional case of positive operators: Thompson's part metric

In the case of $\dim(E) = +\infty$, $\mathbb{P}(E)$ does not have Riemannian structures, so the manifold $\mathbb{P}(E)$ cannot be a $CAT(0)$-space, thus the Karcher and Wasserstein means Λ are not known to admit definitions as minimizers of $C(X) = \int_{\mathbb{P}(E)} d^2(X, A) d\mu(A)$.
However there is another metric:
For $A, B \in \mathbb{P}(E)$ let $d_\infty(A, B) := \|\log(A^{-1/2}BA^{-1/2})\|$.

Theorem (Thompson 1963, Proc. AMS)

The space $(\mathbb{P}(E), d_\infty)$ *is a complete metric space and* $d_\infty(A, B) = \log\max\{M(A \setminus B), M(B \setminus A)\}$ *where* $M(A \setminus B) := \inf\{\beta > 0 : B \leq \beta A\}$.

Infinite dimensional Karcher mean

Let $\mathcal{P}^1(\mathbb{P}(E))$ denote the τ-additive Borel probability measures on $(\mathbb{P}(E), d_\infty)$ s.t. $\int_{\mathbb{P}(E)} d_\infty(X, A) d\mu(A) < \infty$ for $X \in \mathbb{P}(E)$. Note that: τ-additive $\Leftrightarrow \mu(\mathrm{supp}(\mu)) = 1$.
Let $\log_X(A) := X^{1/2} \log \left(X^{-1/2} A X^{-1/2} \right) X^{1/2}$.

Theorem (Karcher mean; Lim-P 2021, Lawson 2018)

For all $\mu \in \mathcal{P}^1(\mathbb{P}(E))$ there exists a unique solution $X \in \mathbb{P}(E)$ of

$$\int_{\mathbb{P}(E)} \log_X(A) d\mu(A) = 0$$

denoted by $\Lambda(\mu)$, which for all $\nu \in \mathcal{P}^1(\mathbb{P}(E))$ satisfies

$$d_\infty(\Lambda(\mu), \Lambda(\nu)) \leq W_1(\mu, \nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \int_{X \times X} d_\infty(a, b) d\gamma(a, b).$$

Generalized Karcher and operator means

Let $\mathcal{P}^0(\mathbb{P}(E))$ denote the set of Borel probability measures with bounded support with respect to d_∞ and let $\mathcal{L} := \{f : (0,\infty) \mapsto \mathbb{R}$ is operator monotone, $f(1) = 0, f'(1) = 1\}$. Notice that $\log(x), x - 1, 1 - x^{-1} \in \mathfrak{L}$.

Corollary (P 2016, Adv. Math.)

Let $f \in \mathfrak{L}$ and $\sigma \in \mathcal{P}^0(\mathbb{P}(E))$. Then the equation

$$\int_{\mathbb{P}(E)} f_X(A) d\sigma(A) = 0$$

has a unique solution $\Lambda_f(\sigma) := X \in \mathbb{P}$, where $f_X(A) = X^{1/2} f\left(X^{-1/2} A X^{-1/2}\right) X^{1/2}$.

Λ_f is called a generalized Karcher mean, it satisfies all the reasonable properties required from an operator mean.

Some examples

Given $\sigma = \sum_{i=1}^{n} \frac{1}{n}\delta_{A_i}$,

- The Karcher mean $\Lambda(\sigma)$, $f(x) = \log x$
- The Arithmetic mean $\sum_{i=1}^{n} w_i A_i$, $f(x) = x - 1$
- The Harmonic mean $\left(\sum_{i=1}^{n} w_i A_i^{-1}\right)^{-1}$, $f(x) = 1 - x^{-1}$
- The Matrix Power means $P_t(\sigma)$, $f(x) = \frac{x^t - 1}{t}$ for $t \in [-1, 1]$

One may also consider more general measures $\sigma \in \mathscr{P}(\mathbb{P})$.
Questions:

- When does a unique zero exist in general?
- How can we calculate these zeros?
- What further properties the zeros have?
- What about the Bures-Wasserstein mean?

Order preserving flows on Thompson metric spaces

Theorem (Gaubert and Qu 2014, JDE)

Let $\phi : \mathbb{P}(E) \mapsto \mathbb{S}(E)$ be a locally Lipschitz function. Assume that the flow generated by the ODE

$$\dot{\eta}(t) = \phi(\eta(t)), \quad t \in [0, \infty)$$

preserves $\mathbb{P}(E)$, and is order-preserving: $\eta(0) \leq \chi(0) \Rightarrow \eta(t) \leq \chi(t)$ for all $t > 0$. Then Gaubert's constant is defined as

$$\alpha(\phi) = \sup\{\beta \in \mathbb{R} : D(x^{-1/2}\varphi(x)x^{-1/2})(x) \leq -\beta, \forall x \in \mathbb{P}(E)\}$$

which is equivalent to the largest flow contraction coefficient α, that is

$$d_\infty(\eta(t), \chi(t)) \leq e^{-\alpha t} d_\infty(\eta(0), \chi(0))$$

where $\eta, \chi : [0, \infty) \mapsto \mathbb{P}(E)$ are solutions of the ODE above.

Proposition (Léka-P 2024)

If the flow contraction coefficient $\alpha > 0$, then

$$\phi(x) = 0$$

has a unique solution.

In particular, $\phi(X) = f_X(A)$ and any convex combinations of such have order preserving flows and Gaubert's constant $\alpha > 0$ when $f \in \mathfrak{L}$.

- This make generalized Karcher means suitable for study using discrete-time contracting (possibly stochastic) ODE flows.
- *For our work covered in the talk the strict exponential contraction of the flow itself suffices!*

Resolvent map and iteration

For $\lambda > 0$ define the generalized resolvent $z = J_\lambda(\phi, x)$ as the unique solution of

$$\lambda\phi(z) + \log_z x = 0.$$

Pick $y_0 \in \mathbb{P}(E)$. Define the sequence of resolvent iterations:

$$y_k = J_{1/k}(\phi, y_{k-1}), \quad k = 1, 2, \dots.$$

Theorem (Léka-P 2024)

Then

$$d_\infty(x_0, y_k) \leq O\left(\frac{1}{n^\alpha}\right),$$

where x_0 is the unique solution of the equation $\phi(x) = 0$.

Properties of the resolvent

Theorem (Contraction property, Léka-P 2024)

We have $d_\infty(J_\lambda(\phi, x), J_\lambda(\phi, y)) \leq \frac{1}{1+\alpha\lambda} d_\infty(x, y)$. *In particular, if* $J_\lambda^\mu(z)$ *denote the unique solution of*

$$\lambda \int_{\mathbb{P}(E)} \phi(y, x)\, d\mu(y) + \log_x z = 0$$

for $\mu \in \mathcal{P}(\mathbb{P}(E))$ *and* $\int_{\mathbb{P}(E)} \phi(y, x)\, d\mu(y)$ *has* $\alpha > 0$, *then*

$$d_\infty(J_\lambda^\mu(x), J_\lambda^\nu(y)) \leq \frac{1}{1+\alpha\lambda} d_\infty(x, y) + \frac{\lambda}{1+\alpha\lambda} \| J_\lambda^\nu(y)^{-1/2} \phi(\cdot, J_\lambda^\mu(y)) J_\lambda^\nu(y)^{-1/2} \|_{\mathrm{Lip}} W_1(\mu, \nu).$$

Holbrook's nodice theorem

Given locally Lipschitz functions $\phi_0, ..., \phi_{k-1}$ from the family of functions with common contraction coefficient $\alpha > 0$, we consider

$$\frac{1}{k}\sum_{i=0}^{k-1} \phi_i(\Lambda) = 0$$

with unique solution Λ.

Theorem (Nodice, Léka-P 2024)

The cyclic resolvent iteration scheme

$$s_{n+1} = J_{\frac{1}{n+1}}(\phi_{\overline{n+1}}, s_n)$$

converges to Λ in d_∞ for any initial data s_0 where $\overline{n}$ denotes the residual of n modulo k.

sketch of the proof.

$\frac{1}{nk+i}\phi_i(s_{nk+i}) + \log_{s_{nk+i}} s_{nk+i-1} = 0, \quad i = 1, \dots, k$. Summing up these inequalities for all $1 \leq i \leq k$, we get

$$\sum_{i=1}^{k} \frac{1}{nk+i}\phi_i(s_{nk+i}) + \sum_{i=1}^{k} \log_{s_{nk+i}} s_{nk+i-1} = 0.$$

For the first term, we have

$$\sum_{i=1}^{k} \frac{1}{nk+i}\phi_i(s_{nk+i}) = \sum_{i=1}^{k} \frac{1}{nk}\phi_i(s_{nk+i})$$
$$+ \sum_{i=1}^{k} \left(\frac{1}{nk+i} - \frac{1}{nk}\right)\phi_i(s_{nk+i}) = \frac{1}{nk}\sum_{i=1}^{k}\phi_i(s_{nk+i}) + O\left(\frac{1}{n^2}\right)$$

due to local boundedness of ϕ_i-s.

proof cont.'d

Since ϕ_i-s are locally Lipschitzian in d_∞, we have

$$d_\infty(\phi_i(s_{nk+i}),\phi_i(s_{nk})) \leq L d_\infty(s_{nk+i},s_{nk}) \leq O\left(\frac{1}{n}\right),$$

thus

$$\sum_{i=1}^{k}\frac{1}{nk+i}\phi_i(s_{nk+i}) = \frac{1}{nk}\sum_{i=1}^{k}\phi_i(s_{(n+1)k}) + O\left(\frac{1}{n^2}\right).$$

Furthermore

$$\begin{aligned}\sum_{i=1}^{k}\log_{s_{nk+i}} s_{nk+i-1} &= \sum_{i=1}^{k} s_{nk+i-1} - s_{nk+i} + O\left(\frac{1}{n^2}\right)\\ &= s_{nk} - s_{(n+1)k} + O\left(\frac{1}{n^2}\right)\\ &= \log_{s_{(n+1)k}} s_{nk} + O\left(\frac{1}{n^2}\right).\end{aligned}$$

proof cont.'d

Therefore

$$\frac{1}{nk}\sum_{i=1}^{k}\phi_i(s_{(n+1)k}) + \log_{s_{(n+1)k}} s_{nk} = O\left(\frac{1}{n^2}\right).$$

Introduce the function ψ_n on $\mathbb{P}$:

$$\psi_n\colon x \mapsto \frac{1}{nk}\sum_{i=1}^{k}\phi_i(x) + \log_x \Lambda.$$

Note hat $\psi_n(x) = 0$ if (and only if) $x = \Lambda$ and $\alpha(\psi_n) \geq 1 + \frac{\alpha}{n}$.
Hence

$$\psi_n(s_{(n+1)k}) = \log_{s_{(n+1)k}} \Lambda - \log_{s_{(n+1)k}} s_{nk} + O\left(\frac{1}{n^2}\right).$$

proof cont.'d

Then the exponential contraction of the flow of ψ_n and EMI imply

$$
\begin{aligned}
d_\infty(\Lambda, s_{(n+1)k}) &\leq \frac{1}{1+\frac{\alpha}{n}} \left\| s_{(n+1)k}^{-1/2} \psi_n(s_{(n+1)k}) s_{(n+1)k}^{-1/2} \right\| \\
&\leq \frac{1}{1+\frac{\alpha}{n}} \left\| s_{(n+1)k}^{-1/2} (\log_{s_{(n+1)k}} \Lambda - \log_{s_{(n+1)k}} s_{nk}) s_{(n+1)k}^{-1/2} \right\| \\
&\quad + O\left(\frac{1}{n^2}\right) \\
&\leq \frac{1}{1+\frac{\alpha}{n}} d_\infty(\Lambda, s_{nk}) + O\left(\frac{1}{n^2}\right),
\end{aligned}
$$

It follows that $d_\infty(\Lambda, s_{nk}) \to 0$ if $n \to \infty$. Also
$d_\infty(s_{nk}, s_{nk+i}) = O(1/n)$ if $1 \leq i \leq k$, thus $d_\infty(\Lambda, s_n) \to 0$. □

Sturm's law of large numbers

Let $\Lambda_\phi(\mu)$ denote the unique solution of

$$\int_{\mathbb{P}(E)} \phi(y,x)\, d\mu(y) = 0$$

for $\mu \in \mathcal{P}(\mathbb{P}(E))$ and assume that $\phi(y,x)$ has contraction coefficient $\alpha > 0$ for each $y \in \mathbb{P}(E)$.

Theorem (slln, Léka-P 2024)

Assume that for each $x \in \mathbb{P}$, there exists an $\varepsilon > 0$ such that the family of functions $\left\{\left\|s^{-1/2}\phi(s,y)s^{-1/2}\right\|\right\}_{s\in B(x,\varepsilon)}$ admits a dominating function $g_{x,\varepsilon} \in L^1(\mu)$. Let $\{Y_n\}_{n\in\mathbb{N}^+}$ be i.i.d. with law μ. Then the stochastic resolvent iteration scheme

$$s_{n+1} = J_{\frac{1}{n+1}}^{\delta_{Y_{n+1}}}(s_n)$$

converges a.s. to $\Lambda_\phi(\mu)$ in d_∞ for any initial data s_0.

Further properties of $\Lambda_\phi(\cdot)$

Proposition (W_1-continuity, Léka-P 2024)

We have

$$d_\infty(\Lambda_\phi(\mu),\Lambda_\phi(\nu)) \leq \frac{1}{\alpha}\left\|\int_{\mathbb{P}(E)}\Lambda_\phi(\nu)^{-1/2}\phi(a,\Lambda_\phi(\nu))\Lambda_\phi(\nu)^{-1/2}d(\mu-\nu)(a)\right\|,$$

that is W_1-**continuity of** $\Lambda_\phi(\cdot)$, *assuming that* $\int_{\mathbb{P}(E)}\|\Lambda_\phi(\nu)^{-1/2}\phi(a,\Lambda_\phi(\nu))\Lambda_\phi(\nu)^{-1/2}\|d\sigma(a)<\infty$ *for* $\sigma\in\{\mu,\nu\}$.

Remark

In the defining equation of J_λ *we can even exchange the penalty term* $\log_x z$ *with* $\phi(z,x)$ *and get similar results.*

Applications to operator means

The (non-commutative) perspectives of f are given by the maps $x \mapsto f_x(z) := x^{1/2} f(x^{-1/2} z x^{-1/2}) x^{1/2}$ for any $f \in \mathfrak{L}$ for which he have $1 - x^{-1} \leq f(x) \leq x - 1$. We now choose

$$\phi(z, x) := f_x(z)$$

Theorem (Léka-P 2024)

Let $f \in \mathfrak{L}$, $\mu \in \mathcal{P}(\mathbb{P})$ such that $\int_{\mathbb{P}} \|f_y(z)\| \, d\mu(z) < \infty$, for all $y \in \mathbb{P}(E)$. Then

$$\dot{x}(t) = \int_{\mathbb{P}} f_{x(t)}(z) \, d\mu(z)$$

generates an order-preserving flow and it has a unique stationary point $\Lambda_f(\mu)$. Furthermore, it has an exponential contraction rate $\alpha_c > 0$ on any closed ball $\overline{B}(I, c) = \{y \in \mathbb{P} : d_\infty(I, y) \leq c\}$.

Corollary (Léka-P 2024)

Under the assumption $\infty > \int_{\mathbb{P}} \|z^{\pm 1}\| \, d\mu(z) \left(\leq \int_{\mathbb{P}} e^{d_\infty(z,I)} \, d\mu(z)\right)$ *we have*

$$\left(\int_{\mathbb{P}} y^{-1} \, d\mu(y)\right)^{-1} \leq \Lambda_f(\mu) \leq \int_{\mathbb{P}} y \, d\mu(y).$$

Let $M_t^f(a,b) := \Lambda_f((1-t)\delta_a + t\delta_b)$ for $t \in [0,1]$ and $a, b \in \mathbb{P}$.

Theorem (slln no. II, Léka-P 2024)

Let $Y_1, Y_2, \ldots$ *be i.i.d. with law* μ *satisfying* $\int_{\mathbb{P}} \|z^{\pm 1}\| \, d\mu(z) < \infty$. *Then the stochastic iteration* $s_0 \in \mathbb{P}$ *and* $s_{n+1} = M^f_{\frac{1}{n+2}}(Y_{n+1}, s_n)$ *converges a.s. to* $\Lambda_f(\mu)$ *with respect to* d_∞.

Nodice also follows with M_t^f as the resolvent. Slln for the arithmetic, geometric (Karcher), harmonic, (matrix) power means follows as special cases.

Relaxing the integrability conditions

Theorem (slln no. III, Léka-P 2026)

Let $Y_1, Y_2, \ldots$ be i.i.d. with law μ satisfying $\int_{\mathbb{P}} \|f(z)\|^2 \, d\mu(z) < \infty$. Then the stochastic iteration $s_0 \in \mathbb{P}$ and $s_{n+1} = J^{\delta_{Y_{n+1}}}_{\frac{1}{n+1}}(s_n)$ converges a.s. to $\Lambda_f(\mu)$ with respect to d_∞.

Theorem (slln no. IV, Léka-P 2026)

Let $Y_1, Y_2, \ldots$ be i.i.d. with law μ satisfying $\int_{\mathbb{P}} \|f(z)\|^2 \, d\mu(z) < \infty$ for $f(x) \leq \log(x)$. Then the stochastic iteration $s_0 \in \mathbb{P}$ and $s_{n+1} = M^f_{\frac{1}{n+2}}(Y_{n+1}, s_n)$ converges a.s. to $\Lambda_f(\mu)$ with respect to d_∞.

It seems possible to relax to $\int_{\mathbb{P}} \|f(z)\| \log(\|f(z)\|) \, d\mu(z) < \infty$. We believe $\int_{\mathbb{P}} \|f(z)\| \, d\mu(z) < \infty$ should be enough to conclude and in slln no. IV. $f(x) \leq \log(x)$ should be superfluous.

The Bures-Wasserstein mean

Let $\mathbb{A} = (A_1, A_2, \ldots, A_m) \in \mathbb{P}^m$ and $\omega = (w_1, w_2, \ldots, w_m) \in \Delta_m$ with $\alpha I \leq A_j \leq \beta I$ for all j and some $\alpha, \beta > 0$. We look at the stationary point equation:

$$\phi(t, X) := \sum_{i=1}^{n} w_i \left(A_i^t \# X^{-1} \right) - I$$

when $t = 1$ this agrees with the Euclidean gradient of the minimization problem for matrices and satisfies

$$\langle D_2 \phi(t, X)(Y), Y \rangle \geq \frac{\alpha^{3t}}{2\beta^{4t}} \langle Y, Y \rangle$$

with respect to Hilbert-Schmidt $Y^* = Y$. From this we obtain:

The Bures-Wasserstein mean

Theorem (Kim-Mer-P 2026)

The equation

$$\Phi(X) := \sum_{i=1}^{m} w_i A_i \# X^{-1} - I = 0$$

has a unique positive definite solution $X \in \mathbb{P}$ defining the Bures-Wasserstein mean $\Omega(\sum_{i=1}^{m} w_i \delta_{A_i}) = \Omega(\omega, \mathbb{A}) = X$.

The *logarithmic norm* of a linear operator A with respect to an operator norm $\|\cdot\|$ on a Banach space is defined and satisfying

$$\mu_{\|\cdot\|}(A) := \sup_{t>0} \frac{\log \left\| e^{tA} \right\|}{t} = \lim_{t \to 0+} \frac{\log \left\| e^{tA} \right\|}{t} = \lim_{t \to 0+} \frac{\|I + tA\| - 1}{t}$$

which describes the largest growth rate of the generated semigroup.

Lemma (Kim-Mer-P 2026)

Given $\epsilon > 0$ *and* $0 < \alpha I \leq X, A_j \leq \beta I$ *with* $X, A_j \in \mathbb{P}$, $\|v\|_{\Re\sigma(-D\Phi(X))+\epsilon} := \int_0^\infty \|e^{-tD\Phi(X)}v\|e^{-t(\Re\sigma(-D\Phi(X))+\epsilon)}dt$ *is an equivalent norm on* $S(\mathcal{H})$ *s.t. its logarithmic norm satisfies*

$$\mu_{\|\cdot\|_{\Re\sigma(-D\Phi(X))+\epsilon}}(-D\Phi(X)) \leq \Re\sigma(-D\Phi(X)) + \epsilon.$$

Define the Banach-Finsler distance:

$$d_\epsilon(A, B) := \inf_{\gamma\in\mathcal{C}^1:\gamma(0)=A,\gamma(1)=B} \int_0^1 \|\dot{\gamma}(t)\|_{\Re\sigma(-D\Phi(\gamma(t)))+\epsilon}dt.$$

Theorem (Kim-Mer-P 2026)

Let $u_i'(t) = -\Phi(u_i(t)), u_i(0) \in \mathbb{P}$. *Then, for all* $t \geq 0$, *we have*

$$d_\epsilon(u_1(t), u_2(t)) \leq e^{(\sup\{\Re\sigma(-D\Phi(X)):X\in S\}+\epsilon)t} d_\epsilon(u_1(0), u_2(0)).$$

Let $\{\lambda_k\}_{k\geq 1}$ be a positive sequence with $\sum_{k=1}^{\infty} \lambda_k = +\infty$ and $\sum_{k=1}^{\infty} \lambda_k^2 < +\infty$.

Proposition (Forward Euler step method, Kim-Mer-P 2026)

Let $X_* \in \mathbb{P}$ be such that $\Phi(X_*) = 0$,
$a := \sup\{\Re\sigma(-D\Phi(X)) : X \in [\alpha I, \beta I]\} + \epsilon$ and let

$$X_{k+1} = X_k - \lambda_{k+1}\Phi(X_k) + O\left(\lambda_{k+1}^2\right)$$

recursively with $X_1 \in \mathbb{P}$. Then

$$d_\epsilon(X^*, X_{k+1}) \leq (1 + a\lambda_{k+1})\, d_\epsilon(X^*, X_k) + O\left(\lambda_{k+1}^2\right)$$

and $d_\epsilon(X^*, X_k) \to 0$ as $k \to \infty$ as long as $\sum_{k=1}^{\infty} \lambda_k^2$ is not exceeding a certain a priori threshold.

Nodice theorem

For each $Z \in [\alpha I, \beta I]$ and $h > 0$, the resolvent $J_h^Z : [\alpha I, \beta I] \to [\alpha I, \beta I]$ is defined by

$$J_h^Z(X) = Y \quad \iff \quad 0 = \Psi_Z(X) + \frac{1}{h}(Y - X) + O(h^2),$$

where $\Psi_Z(X) := (I - (Z \# X^{-1}))$.

Theorem (No-dice theorem, Kim-Mer-P 2026)

For any $s_0 \in [\alpha I, \beta I]$, define the cyclic resolvent iteration

$$s_{n+1} = J_{1/(N+n+1)}^{A_{i_n}}(s_n), \qquad n \geq 0,$$

where $i_n \in \{1, \ldots, m\}$ is determined by $i_n \equiv n + 1 \pmod{m}$ and N is large enough a priori calculated. Then $s_n \longrightarrow \Omega((1/m, \ldots, 1/m), \mathbb{A})$ in the metric d_ε.

Strong law for the Wasserstein mean

Let μ be a Borel probability measure with $\mathrm{supp}(\mu) \subseteq ([\alpha I, \beta I], \|\cdot\|)$ and $\mu(\mathrm{supp}(\mu)) = 1$.

Theorem (Kim-Mer-P 2026)

Given a $\mathbb{P}$-valued random variable Y with law μ,

$$0 = \mathbb{E}_Y \Psi_Y(X) = \int_{\mathbb{P}} \Psi_A(X) d\mu(A)$$

has a unique solution X in $\mathbb{P}$ denoted by $\Omega(\mu)$.

Theorem (Strong law of large numbers, Kim-Mer-P 2026)

Let $\{Y_k\}_{n\in\mathbb{N}}$ be i.i.d. with law μ. For any $s_0 \in [\alpha I, \beta I]$, define

$$s_{n+1} = J_{1/(N+n+1)}^{Y_{n+1}}(s_n), \qquad n \geq 0,$$

for a fixed suitable $N > 0$. Then $s_n \longrightarrow \Omega(\mu)$ a.s. in the metric d_ε.

Problems to work on

1. In any of the slln eliminate any unnecessary higher moment conditions.
2. Extend the above theory to Finsler spaces for nonlinear monotone operators.
3. When there is an slln available establish a CLT.
4. Formulate an slln for operator monotone free functions of probability measures on positive operators.

Thank you for your attention!

Z. Léka and M. P., *Strong law of large numbers for generalized operator means*, Adv. Math. 457 (2024), 109933.

S. Gaubert and Z. Qu, The contraction rate in Thompson's part metric of order-preserving flows on a cone – Application to generalized Riccati equations, J. Diff. Equ. 256:8 (2014), pp. 2902–2948.

Y. Lim and M. P., *Strong law of large numbers for the L^1-Karcher mean*, J. Func. Anal. 279:7 (2020), 108672.

Y. Lim and M. P., *Existence and uniqueness of the L^1-Karcher mean*, Adv. Math. 376 (2021), 107435.

M. P., *Operator means of probability measures and generalized Karcher equations*, Adv. Math. 289 (2016), pp. 951–1007.

Thank you for your kind attention!