OPTIMAL TRANSPORT, HEAT FLOW, AND RICCI CURVATURE ON METRIC MEASURE SPACES

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Abstract

We present a brief survey on the theory of metric measure spaces with synthetic lower Ricci bounds, initiated by the author and by Lott/Villani, and developed further by Ambrosio/Gigli/Savare and by many others. Particular emphasis will be given to recent breakthroughs concerning the local structure of RCD-spaces by Mondino/Naber and by Brue/Semola and to rigidity results. For instance, given an arbitrary RCD(N-1,N)-space (X, d, m), then

$$\int\int \cos d(x,y)\, dm(x)\, dm(y) \leq 0$$

if and only if N is an integer and (X, d, m) is isomorphic to the N-dimensional round sphere. Moreover, we study the heat equation on time-dependent metric measure spaces and its dual as gradient flows for the energy and for the Boltzmann entropy, resp. Monotonicity estimates for transportation distances and for squared gradients will be shown to be equivalent to the so-called dynamical convexity of the Boltzmann entropy on the Wasserstein space which is the defining property of super-Ricci flows. Moreover, we show the equivalence with the monotone coupling property for pairs of backward Brownian motions as well as with log Sobolev, local Poincare and dimension free Harnack inequalities.

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