Minimal curves in $\mathcal{U}(n)$ and $\mathcal{G}l(n)^+$ with respect to the spectral and the trace norms

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Consider the Lie group of $n \times n$ complex unitary matrices $\mathcal{U}(n)$ endowed with the bi-invariant Finsler metric given by the spectral norm,

$$\|X\|_{_U} = \|U^*X\|_\infty = \|X\|_\infty$$

for any X tangent to a unitary operator U. Given two points in $\mathcal{U}(n)$, in general there exists infinitely many curves of minimal length. In this talk we will provide a complete description of such curves. As a consequence of this description, we conclude that there is a unique curve of minimal length between U and V if and only if the spectrum of U^*V is contained in a set of the form $\{e^{i\theta}, e^{-i\theta}\}$ for some $\theta \in [0, \pi)$. This is similar to the result obtained by Lim in [6] in the case of positive operators endowed with the Thompson metric. Let's recall that the Grassmannian can be modeled as a submanifold of the unitary group (identifying a subspace with the associated orthogonal symmetry). Using this idea we also describe all minimal curves connecting two projections P and Q such that $\|P - Q\|_\infty < 1$.

Now consider the cone of $n \times n$ positive invertible matrices $\mathcal{G}l(n)^+$ endowed with the bi-invariant Finsler metric given by the trace norm,

$$\|X\|_{_{1,\,A}} = \|A^{-1/2}XA^{-1/2}\|_1$$

for any X tangent to $A \in \mathcal{G}l(n)^+$. In this context, given two points $A, B \in \mathcal{G}l(n)^+$, there also exists infinitely many curves of minimal length. In this case we get the characterization by lifting the problem to the space of hermitian matrices $\mathcal{H}(n)$. So, firstly we provide a characterization of the minimal paths joining $X, Y \in \mathcal{H}(n)$, if the length of a curve $\alpha : [a, b] \to \mathcal{H}(n)$ is measure by

$$L(\alpha) = \int_a^b \|\dot{\alpha}(t)\|_1 \, dt.$$

Once the characterization of the minimal curves is given for $\mathcal{H}(n)$, the characterization in $\mathcal{G}l(n)^+$ can be obtained using the Exponential Metric Increasing property. On the other hand for $\mathcal{U}(n)$ the above lifting argument also works in one direction. Indeed, using the same idea as in the case of $\mathcal{G}l(n)^+$, we prove that a minimal curve in $\mathcal{H}(n)$ leads to a minimal curve in $\mathcal{U}(n)$ by means of the exponential map. So our characterization in $\mathcal{H}(n)$ give us a way to construct minimal paths in the group of unitary matrices $\mathcal{U}(n)$.

In [6] Lim also studied the sets of midpoints

$$\mathcal{M}_t(A, B) = \big\{C \in \mathcal{G}l(n)^+ : d_\infty(A, C) = t\, d_\infty(A, B),\ d_\infty(C, B) = (1 - t)\, d_\infty(A, B)\big\}.$$

In this talk we will study the set of midpoints in all the previous contexts. We will prove that this set is geodesically convex for $\mathcal{G}l(n)^+$ for any unitarily invariant norm. This was already prove in [6] but we will use a different technique. The same idea turns out to work to prove that the set of midpoints set is geodesically convex for $\mathcal{H}(n)$ endowed with any unitarily invariant norm and for the spectral norm between two unitary matrices U and V provided $\|U-V\|_\infty < 1$.

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